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Fixed Point Results in Modular S-Metric Space

B. Y. Aiyetan^{a*} and K. Rauf^b

^aDepartment of Mathematics, Federal University Oye-Ekiti, Ekiti, Nigeria.

^bDepartment of Mathematical, University of Ilorin, Ilorin, Nigeria.

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this work, some definitions, properties and propositions in modular S-metric space were given, existence, uniqueness and coincidence point of a fixed point results in the space were also provided using some general contractive conditions defined in modular S-metric space settings. Our results shows that existence and uniqueness of $[L, \phi]$ contraction is possible and also the coincidence of a fixed point for a specific rational contraction can be determined which improves on some results in the literature.

Keywords: Modular S-metric space; coincidence point; modular convergent; modular cauchy.

2010 Mathematics Subject Classification: 32H50.

*Corresponding author: E-mail: barnabas.aiyetan@fuoye.edu.ng, bornjery@gmail.com;

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Nomenclature

W_λ^s	Modular S-metric
λ	Time taking to cover a distance between two points

1 Introduction

The concept of distance space was first studied by Frechet (1905), where he defined the gap between two points as a metric d on a non-empty set X satisfying some certain conditions. As a result of his work, many researchers have come up with several generalization of metric space, among them is Gähler Gähler (1963) who introduced the notion of 2-metric space which is a generalization of ordinary metric space.

In Dhage (1992) introduced a new structure of a generalized metric space called D-metric space; also in 2003, Mustafa and Sims (2006) defined another generalization of metric space, and claimed that properties of D-metric space were wrong and as a result, they gave a new properties of a distance function called G-metric space (Rauf et al., 2019; Rauf and Aiyetan, 2018). In 2012, Sedghi et al. (2012) introduced S-metric spaces and gave some properties. They proved the fixed point theorem for a self-mapping on a complete S-metric space.

Gopi et al. (2023) proved common fixed point results for monotone asymptotic pointwise ρ - nonexpansive semigroups in modular function spaces. Thus, utilizing the monotonicity of semigroups, there findings generalized and extend some prominent recent results of the literature in the setting of modular function spaces (Azadifar et al., 2013a; Kozłowski, 1988).

The idea of modular space was conceived by Nakano (1950) in which conform with the theory of order spaces.

Musielak and Orlicz (1959) build on this idea of modular space by proving some fixed point theorems. Chistyakov (2006) in introduced the notion of modular metric on an ordinary set and the corresponding modular space which is more general than a metric space.

In line with the concept of Chistyakov (2006, 2011), many researchers have come up with several generalization of modular metric spaces, such as Azadifar et al. (2013b) in who gave a generalization of modular metric space to develop a space called modular G-metric space. Several results of fixed point were gotten using contractive mappings defined on modular G-metric spaces Sedghi et al. (2018).

Arockiam and John (2024) obtained common fixed point theorems for three mappings of contractive type in the setting of generalized modular metric spaces. There results generalized many results available in the literature including common fixed point theorems.

In the present paper, we introduce the concept of modular S-metric space, define modular S-metric space, prove some properties of modular S-metric space, existence and uniqueness as well as coincidence point of a modular S-metric spaces were determined.

2 Preliminary

Here, we shall define the Modular S-metric space based on the concept of modular metric space defined in Chistyakov (2006) and that of modular G-metric space define in Azadifar et al. (2013b).

Definition 2.1: Let X be a nonempty set, and $W_\lambda^s : (0, \infty) \times X \times X \times X \rightarrow [0, \infty]$ be a function satisfying:

1. $W_\lambda^s(x, y, z) \geq 0$ for all $x, y, z \in X$ and $\lambda > 0$;

2. $W_\lambda^s(x, y, z) = 0$ for all $x, y, z \in X$ and $\lambda > 0$ if and if $x = y = z$; and
3. $W_{\lambda+\mu+\gamma}^s(x, y, z) \leq W_\lambda^s(x, x, a) + W_\mu^s(y, y, a) + W_\gamma^s(z, z, a)$ for all $x, y, z, a \in X$ and $\lambda, \mu, \gamma > 0$.
Then the function $w_\lambda^s(\cdot, \cdot)$ is called a modular S -metric space on X and the pair (X, W_λ^s) is called a modular S - metric space and $W_\lambda^s, W_\mu^s, W_\gamma^s$ are velocities with respect to time λ, μ and γ respectively.

Example 2.1: Let $X = [0, \infty)$ be the interval of nonnegative real numbers and let W_λ^s be defined by

$$W_\lambda^s(x, y, z) = \begin{cases} 0, & \text{if } x=y=z. \\ \max(x,y,z), & \text{otherwise} \end{cases}$$

Then W_λ^s is a complete W_λ^s -metric on X .

In this section, we introduce some definitions and notion of W_λ^s -convergent sequence and W_λ^s - Cauchy sequence.

Definition 2.2: A sequence $\{x_n\}$ in X is said to be modular S convergent to $x \in X$ if $W_\lambda^s(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, that is for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $W_\lambda^s(x_n, x_n, x) < \epsilon$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$

Definition 2.3: A sequence $\{x_n\}$ in X is said to be modular S Cauchy sequence if $W_\lambda^s(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$, that is for every $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $W_\lambda^s(x_n, x_n, x_m) < \epsilon$.

Definition 2.4: The modular S-metric space (X, W_λ^s) is called complete if every modular S-Cauchy sequence in it is a modular S-convergent sequence.

Definition 2.5: Let (X, W_λ^s) be a modular S-metric space, and $\{x_n\}$ be a sequence of points of X . A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n, m \rightarrow \infty} W_\lambda^s(x, x_n, x_n)$ and one say the sequence $\{x_n\}$ is a modular S-convergent to x .

Thus, that if $x_n \rightarrow 0$ in a modular S-metric space (X, W_λ^s) , then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $W_\lambda^s(x, x_n, x_m) < \epsilon$, for all $n, m, \in N$.

3 The Heart of the Matter

Definition 3.1 There exists a real number $L \geq 0$ and a strict comparison function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ for all $x, y, z \in X$ such that

$$W_\lambda^s(Tx, Ty, Tz) \leq LW_\lambda^s(x, Tx, Tx) + \phi W_\lambda^s(x, y, z) \tag{1}$$

where $\lambda > 0$ and $0 \leq L + \phi < 1$.

Definition 3.2 There exists a real number a, b, c , where $0 \leq a + b + c < 1$ such that for each $x, y, z \in X; \lambda > 0$

$$W_\lambda^s(Tx, Ty, Tz) \leq aW_\lambda^s(x, Tx, Tx) + W_\lambda^s(y, Ty, Ty) + cW_\lambda^s(x, y, z) \tag{2}$$

Definition 3.3 There exists a real number α, β where $0 \leq \alpha + \beta < 1$ such that for each $x, y, z \in X; \lambda > 0$ we have

$$W_\lambda^s(Tx, Ty, Tz) \leq \alpha \frac{W_\lambda^s(x, Tx, Tx)W_\lambda^s(y, Ty, Ty)}{W_\lambda^s(x, y, z)} + \beta W_\lambda^s(x, y, z) \quad (3)$$

Proposition 3.1: Let (X, W_λ^s) be a modular S-metric space then the following are equivalent:

1. $\{x_n\}$ is w_λ^s -convergent to x for a function $F : X \rightarrow X$;
2. $W_\lambda^s(x_n, x_n, x) = 0$; as $n \rightarrow \infty$;
3. $W_\lambda^s(x_n, x, x) = 0$; as $n \rightarrow \infty$; and
4. $W_\lambda^s(x_n, x_m, x) = 0$; as $n, m, \rightarrow \infty$.

Lemma 3.1: Let (X, W_λ^s) be a modular S-metric space. If $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$ then $W_\lambda^s(x_n, x_n, y_n) \rightarrow W_\lambda^s(x, x, y)$ as $n \rightarrow \infty$

Lemma 3.2: Let T be a self mapping on a modular S-metric space (X, W_λ^s) . Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X , $x_n \rightarrow x$ implies $Tx_n \rightarrow Tx$ as $n \rightarrow \infty$.

Lemma 3.3: In a modular S-metric space, we have $W_\lambda^s(x, x, y) = W_\lambda^s(y, y, x)$ for all $x, y \in X \lambda > 0$.

Proof. By the third condition of modular S-metric space, we have $W_\lambda^s(x, y, y) \leq W_\lambda^s(x, x, a) + W_\lambda^s(y, y, a) + W_\lambda^s(z, z, a)$

$$\begin{aligned} W_\lambda^s(x, x, y) &\leq W_\lambda^s(x, x, x) + W_\lambda^s(x, x, x) + W_\lambda^s(y, y, x) \\ &\leq W_\lambda^s(y, y, x) \\ W_\lambda^s(x, x, y) &\leq W_\lambda^s(y, y, x) \end{aligned}$$

Similarly

$$\begin{aligned} W_\lambda^s(y, y, x) &\leq W_\lambda^s(y, y, y) + W_\lambda^s(y, y, y) + W_\lambda^s(x, x, y) \\ &\leq W_\lambda^s(x, x, y) \\ W_\lambda^s(y, y, x) &\leq W_\lambda^s(x, x, y) \end{aligned}$$

Therefore, $W_\lambda^s(y, y, x) = W_\lambda^s(x, x, y)$ \square

Proposition 3.2: The limit of a W_λ^s -convergent sequence in a W_λ^s -metric space is unique.

Proof. Let (X, W_λ^s) be a W_λ^s -metric space and let $\{x_n\} \subseteq X$ be a sequence that converges at the same time to $x \in X$ and to $y \in X$. We claim that $W_\lambda^s(x, y, y) < \epsilon$ for all $\epsilon > 0$. Let $\epsilon > 0$ be arbitrary, there exist natural numbers $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned} W_\lambda^s(x_n, x_m, x) &\leq \frac{\epsilon}{3} \text{ for all } n, m \geq n_1 \\ W_\lambda^s(x_n, x_m, y) &\leq \frac{\epsilon}{3} \text{ for all } n, m \geq n_2 \end{aligned}$$

Let $n_0 = \max(n_1, n_2)$. Then by the third axioms of W_λ^s -metric space we have

$$\begin{aligned} W_\lambda^s(x, x, y) &\leq W_\lambda^s(x, x, x_n) + W_\lambda^s(x, x, x_n) + W_\lambda^s(y, y, x_n) \\ &\leq 2W_\lambda^s(x, x, x_n) + W_\lambda^s(y, y, x_n) \\ &= 2\frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Consequently, we have that $W_\lambda^s(x, x, y) = 0$ and by axiom two of W_λ^s -metric space, we conclude that $x = y$. \square

Proposition 3.3: Every convergent sequence in a W_λ^s S-metric space is a Cauchy sequence.

Proof. Let (X, W_λ^s) be a W_λ^s S-metric space and let $\{x_n\} \subseteq X$ be a sequence that converges to $x \in X$. Let $\epsilon > 0$ be arbitrary. By definition, there exist $n_0 \in \mathbb{N}$ such that

$W_\lambda^s(x_n, x_n, x) \leq \frac{\epsilon}{3}$ for all $n, m \geq n_0$ By the third axiom of modular S-metric space we have that for all $n, m, k \geq n_0$

$$W_\lambda^s(x_n, x_m, x_k) \leq W_\lambda^s(x_n, x_n, x) + W_\lambda^s(x_m, x_m, x) + W_\lambda^s(x_k, x_k, x) \tag{4}$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \tag{5}$$

$$= \epsilon \tag{6}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in (X, W_λ^s) \square

We state our main results as follows.

Theorem 3.1 Let W_λ^s be a complete S-metric space. Let $T : X \rightarrow X$ be a continuous mapping of modular S-contraction where there exists a real number $L \geq 0$ and a strict comparison function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ for all $x, y, z \in X$ such that

$$W_\lambda^s(Tx, Ty, Tz) \leq LW_\lambda^s(x, Tx, Tx) + \phi W_\lambda^s(x, y, z) \tag{7}$$

where $\lambda > 0$ and $0 \leq L + \phi < 1$. Then T has a unique fixed point p in X .

Proof. If $x = x_n; y = z = y_n$

Let $x_n \neq x_{n+1}$ and $x_{n+1} = y_n$
then

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) = W_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \tag{8}$$

Let $x_0 \in X$ be an arbitrary point in X and $x_{n+1} = Tx_n$ be a Picard iteration. Then for $n = 0, x_1 = Tx_0$.

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) = W_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \tag{9}$$

if $n=1$

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq LW_\lambda^s(x_0, Tx_0, Tx_0) + \phi W_\lambda^s(x_0, Tx_0, Tx_0) \quad (10)$$

$$= LW_\lambda^s(x_0, x_1, x_1) + \phi W_\lambda^s(x_0, x_1, x_1) \quad (11)$$

$$\leq [L + \phi]W_\lambda^s(x_0, x_1, x_1) \quad (12)$$

for $n=2$

$$W_\lambda^s(x_2, x_3, x_3) \leq LW_\lambda^s(x_1, Tx_0, Tx_0) + \phi W_\lambda^s(x_0, Tx_0, Tx_0) \quad (13)$$

$$\leq [L + \phi]W_\lambda^s(x_1, x_2, x_2) \quad (14)$$

$$\leq [L + \phi]^2 W_\lambda^s(x_0, x_1, x_1) \quad (15)$$

for $n=3$

$$W_\lambda^s(x_3, x_4, x_4) \leq LW_\lambda^s(x_2, x_3, x_3) + \phi W_\lambda^s(x_2, x_3, x_3) \quad (16)$$

$$\leq [L + \phi]W_\lambda^s(x_2, x_3, x_3) \quad (17)$$

$$\leq [L + \phi]^3 W_\lambda^s(x_0, x_1, x_1) \quad (18)$$

iteratively to n we have;

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq [L + \phi]^n W_\lambda^s(x_0, x_1, x_1) \quad (19)$$

Let $\beta = [L + \phi] < 1$

Hence;

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq [L + \phi]^n W_\lambda^s(x_0, x_1, x_1) \quad (20)$$

$$= \beta^n W_\lambda^s(x_0, x_1, x_1) \quad (21)$$

which prove the convergence of the sequence $\{x_n\}$.

Next we show the sequence $\{x_n\}$ is a Cauchy sequence for any positive integer $m, n \in \mathbb{N}$ and $N \in \mathbb{N}$ such that $m > n$.

Then,

$$W_\lambda^s(x_n, x_m, x_m) \leq W_\lambda^s(x_n, x_{n+1}, x_{n+1}) + W_\lambda^s(x_{n+1}, x_{n+2}, x_{n+2}) + W_\lambda^s(x_{n+2}, x_{n+3}, x_{n+3}) \quad (22)$$

$$+ \dots + W_\lambda^s(x_{m-1}, x_m, x_m) \quad (23)$$

$$\leq \beta^n W_\lambda^s(x_0, x_1, x_1) + \beta^{n+1} W_\lambda^s(x_0, x_1, x_1) + \beta^{n+2} W_\lambda^s(x_0, x_1, x_1) + \dots \quad (24)$$

$$\leq (\beta^n + \beta^{n+1} + \beta^{n+2} + \dots) W_\lambda^s(x_0, x_1, x_1) \quad (25)$$

$$W_\lambda^s(x_n, x_m, x_m) \leq \left(\frac{\beta^n}{1 - \beta}\right) W_\lambda^s(x_0, x_1, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (26)$$

Therefore, the sequence $\{x_n\}$ is a W_λ^s Cauchy sequence. Then the sequence is a W_λ^s -complete.

If the limit of the sequence $\{x_n\}$ is defined as p then the contraction becomes. $W_\lambda^s(x_{n+1}, Tp, Tp) \leq LW_\lambda^s(x_n, Tp, Tp) + \phi W_\lambda^s(x_n, Tp, Tp)$.

Taking the limit of the sequence as $n \rightarrow \infty$, and with the fact that W_λ^s continuous by Lemma2.2. Then,

$$W_\lambda^s(x_{n+1}, Tp, Tp) \leq LW_\lambda^s(p, Tp, Tp) + \phi W_\lambda^s(p, Tp, Tp) \tag{27}$$

$$\leq \beta W_\lambda^s(p, Tp, Tp) \tag{28}$$

Since $0 \leq \beta < 1$ hence, $p = Tp$.

Suppose q is also a fixed point of T and $p \neq q$. Then the contraction becomes

$$W_\lambda^s(p, q, q) \leq LW_\lambda^s(p, q, q) + \phi W_\lambda^s(p, q, q) \tag{29}$$

$$W_\lambda^s(p, q, q) \leq \beta W_\lambda^s(p, q, q) < W_\lambda^s(p, q, q) \tag{30}$$

which is only possible if $p = q$ therefore, T has a unique fixed point p in X . □

Theorem 3.2 Let (X, W_λ^s) be a complete modular S-metric space and let $T : X \rightarrow X$ be a continuous modular S-contraction such that, there exists a real number a, b, c $0 \leq a + b + c < 1$ such that for each $x, y, z \in X$; $\lambda > 0$

$$W_\lambda^s(Tx, Ty, Tz) \leq aW_\lambda^s(x, Tx, Tx) + bW_\lambda^s(y, Ty, Ty) + cW_\lambda^s(x, y, z) \tag{31}$$

Then T has a unique fixed point in X .

Proof. If $x = x_n$; $y = z = y_n$ Let $x_n \neq x_{n+1}$ and $x_{n+1} = y_n$

Then,

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) = W_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \tag{32}$$

$$\leq aW_\lambda^s(x_{n-1}, x_n, x_n) + bW_\lambda^s(x_n, x_{n+1}, x_{n+1}) + cW_\lambda^s(x_n, x_{n+1}, x_{n+1}) \tag{33}$$

$$= aW_\lambda^s(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) + bW_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \tag{34}$$

$$+ cW_\lambda^s(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) \tag{35}$$

$$\leq a^2W_\lambda^s(x_{n-2}, x_{n-1}, x_{n-1}) + b^2W_\lambda^s(x_{n-1}, x_n, x_n) \tag{36}$$

$$+ c^2W_\lambda^s(x_{n-2}, x_{n-1}, x_{n-1}) \tag{37}$$

Iteratively to n we have

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq a^n W_\lambda^s(x_0, x_1, x_1) + b^n W_\lambda^s(x_1, x_2, x_2) \tag{38}$$

$$+ c^n W_\lambda^s(x_0, x_1, x_1) \tag{39}$$

$$\leq [a^n + b^{n+1} + c^n] W_\lambda^s(x_0, x_1, x_1) \tag{40}$$

let $\beta = [a^n + b^{n+1} + c^n] < 1$

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq \beta^n W_\lambda^s(x_0, x_1, x_1) \tag{41}$$

This proves that the sequence $\{x_n\}$ is W_λ^s -converges.

For every $m > n \in \mathbb{N}$ and $N \in \mathbb{N}$

$$W_\lambda^s(x_n, x_m, x_m) \leq W_\lambda^s(x_n, x_{n+1}, x_{n+1}) + W_\lambda^s(x_{n+1}, x_{n+2}, x_{n+2}) + W_\lambda^s(x_{n+2}, x_{n+3}, x_{n+3}) \quad (42)$$

$$+ \dots + W_\lambda^s(x_{m-1}, x_m, x_m) \quad (43)$$

$$\leq \beta^n W_\lambda^s(x_0, x_1, x_1) + \beta^{n+1} W_\lambda^s(x_0, x_1, x_1) + \beta^{n+2} W_\lambda^s(x_0, x_1, x_1) + \dots \quad (44)$$

$$\leq (\beta^n + \beta^{n+1} + \beta^{n+2} + \dots) W_\lambda^s(x_0, x_1, x_1) \quad (45)$$

$$W_\lambda^s(x_n, x_m, x_m) \leq \left(\frac{\beta^n}{1-\beta}\right) W_\lambda^s(x_0, x_1, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (46)$$

This proves the sequence $\{x_n\}$ is W_λ^s -Cauchy. Hence, $\{x_n\}$ is W_λ^s -complete.

If the limit of the sequence $\{x_n\}$ be defined as p then the contraction becomes.

$W_\lambda^s(x_{n+1}, Tp, Tp) \leq aW_\lambda^s(x_{n-1}, Tp, Tp) + bW_\lambda^s(x_n, Tp, Tp) + aW_\lambda^s(x_{n-1}, Tp, Tp)$ Taking the limit of the sequence at $n \rightarrow \infty$, and with the fact that W_λ^s continuous. Then,

$$W_\lambda^s(p, Tp, Tp) \leq aW_\lambda^s(p, Tp, Tp) + bW_\lambda^s(p, Tp, Tp) + aW_\lambda^s(p, Tp, Tp) \quad (47)$$

$$W_\lambda^s(p, Tp, Tp) \leq \beta W_\lambda^s(p, Tp, Tp) \quad (48)$$

since $0 \leq \beta < 1$ Hence,

$$W_\lambda^s(p, Tp, Tp) \leq \beta W_\lambda^s(p, Tp, Tp) = W_\lambda^s(p, Tp, Tp) \quad (49)$$

$$Tp = p. \quad (50)$$

Since $0 \leq \beta < 1$ Hence, $p = Tp$.

Suppose q is also a fixed point of T and $p \neq q$. Then the contraction becomes

$$W_\lambda^s(p, q, q) \leq aW_\lambda^s(p, q, q) + bW_\lambda^s(p, q, q) + cW_\lambda^s(p, q, q) \quad (51)$$

$$W_\lambda^s(p, q, q) \leq \beta W_\lambda^s(p, q, q) < W_\lambda^s(p, q, q) \quad (52)$$

Since $0 \leq \beta < 1$ implies $p = q$.

therefore, T has a unique fixed point p in X . \square

Here, we consider the coincidence point of a fixed point of modular S-metric space.

Theorem 3.3 Let (X, W_λ^s) be a modular S-metric space and $T, g : X \rightarrow X$ be two self-mapping of W_λ^s -contraction were there exists a real number α, β where $0 \leq \alpha + \beta < 1$ such that for each $x, y, z \in X; \lambda > 0$ we have

$$W_\lambda^s(Tx, Ty, Tz) \leq \alpha \frac{W_\lambda^s(x, Tx, Tx)W_\lambda^s(y, Ty, Ty)}{W_\lambda^s(x, y, z)} + \beta W_\lambda^s(x, y, z) \quad (53)$$

Assume that the following conditions are fulfilled:

1. (X, W_λ^s) is complete;
2. $T(X) \subseteq g(x)$; and
3. g is W_λ^s is continuous and commutes with T .

Then T and g have a unique coincidence point. That is a unique point $p \in X$ such that $gp = Tp = p$ exists.

Proof. Let x_0 be an arbitrary element of X constructing a sequence $\{x_n\}$ defined as follows $gx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$.

Here, we wish show that the sequence $\{x_n\}$ is W_λ^s -convergent.

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) = (Tgx_{n-1}, Tgx_n, Tgx_n) \tag{54}$$

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \leq \alpha \frac{W_\lambda^s(gx_{n-1}, gx_n, gx_n)W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1})}{W_\lambda^s(gx_{n-1}, gx_n, gx_n)} \tag{55}$$

$$+ \beta W_\lambda^s(gx_{n-1}, gx_n, gx_n) \tag{56}$$

$$\leq \alpha W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) + \beta W_\lambda^s(gx_{n-1}, gx_n, gx_n) \tag{57}$$

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) - \alpha W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \tag{58}$$

$$\leq \beta W_\lambda^s(gx_{n-1}, gx_n, gx_n) \tag{59}$$

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \leq \left(\frac{\beta}{1-\beta}\right)W_\lambda^s(gx_{n-1}, gx_n, gx_n) \tag{60}$$

$$= \left(\frac{\beta}{1-\alpha}\right)W_\lambda^s(Tgx_{n-2}, Tgx_{n-1}, Tgx_{n-1}) \tag{61}$$

$$\leq \left(\frac{\beta}{1-\alpha}\right)^2 W_\lambda^s(gx_{n-1}, gx_n, gx_n) \tag{62}$$

iteratively, we have

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \leq \left(\frac{\beta}{1-\alpha}\right)^n W_\lambda^s(gx_0, gx_1, gx_1) \tag{63}$$

let

$$\frac{\beta}{1-\alpha} = k < 1$$

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \leq k^n W_\lambda^s(gx_0, gx_1, gx_1). \tag{64}$$

we conclude that, $W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1})$ converges to $W_\lambda^s(gx_0, gx_1, gx_1)$ as $n \rightarrow \infty$. Then, the sequence $\{x_n\}$ is W_λ^s -convergent.

Now, we shall show that the sequence $\{x_n\}$ is W_λ^s -Cauchy.

$$W_\lambda^s(x_n, x_m, x_m) \leq W_\lambda^s(x_n, x_{n+1}, x_{n+1}) + W_\lambda^s(x_{n+1}, x_{n+2}, x_{n+2}) + W_\lambda^s(x_{n+2}, x_{n+3}, x_{n+3}) \quad (65)$$

$$+ \dots + W_\lambda^s(x_{m-1}, x_m, x_m) \quad (66)$$

$$\leq k^n W_\lambda^s(x_0, x_1, x_1) + k^{n+1} W_\lambda^s(x_0, x_1, x_1) + k^{n+2} W_\lambda^s(x_0, x_1, x_1) + \dots \quad (67)$$

$$\leq (k^n + k^{n+1} + k^{n+2} + \dots) W_\lambda^s(x_0, x_1, x_1) \quad (68)$$

$$W_\lambda^s(x_n, x_m, x_m) \leq \left(\frac{k^n}{1-k}\right) W_\lambda^s(x_0, x_1, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (69)$$

This, confirm the sequence $\{x_n\}$ is W_λ^s -Cauchy.

Here, we show that p is a coincidence point of T and g .

$$W_\lambda^s(gx_{n+1}, gp, gp) = (Tgx_n, Tgp, Tgp) \quad (70)$$

$$\leq \alpha \frac{W_\lambda^s(gx_n, gp, gp) W_\lambda^s(gp, gp, gp)}{W_\lambda^s(gx_n, gp, gp)} + \beta W_\lambda^s(gx_{n+1}, gp, gp) \quad (71)$$

Hence,

$$W_\lambda^s(gx_{n+1}, gp, gp) \leq \beta W_\lambda^s(gx_{n+1}, gp, gp) \quad (72)$$

Since g is W_λ^s -continuous, $\{gx_n\} \rightarrow gp$ also g and T commute, we have $gx_{n+1} = Tgx_n = gTx_n$ for all $n \geq 0$.

letting $n \rightarrow \infty$ and the fact that T is continuous we have,

$$W_\lambda^s(gp, Tp, Tp) \leq \beta W_\lambda^s(gp, gp, gp) = 0$$

Hence, $gp = Tp$

We now show the uniqueness of the coincidence point of T and g . Suppose, there exists another coincidence point $q \in X$ with $q \neq p$.

from our contraction we have;

$$W_\lambda^s(p, q, q) \leq \beta W_\lambda^s(Tp, Tq, Tq) = 0 \quad (73)$$

which is a contradiction since $\beta < 1$.

Hence, the coincidence point of T and g is unique. □

4 Conclusion

Results in modular S-metric have been established, using some generalized contractions defined in modular space, the existence and uniqueness where there exists a real number $L \geq 0$ and a strict comparison function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ for all $x, y, y \in X$ were established, and coincidence point of a fixed point results for Rational-Type contraction were also established.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Authors have declared that no competing interests exist.

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