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A Duality Principle and Concerning Convex Dual Formulation for a Model in Micro-Magnetism

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Authors' contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

This article develops a duality principle applicable to originally non-convex primal variational formulations. More specifically, we establish a convex (in fact concave) dual variational formulation for a model in micro-magnetism. The results are obtained through basic tools of functional analysis, calculus of variations, duality and optimization theory in infinite dimensional spaces. It is worth highlighting the dual functional obtained is concave in its concerning main variables, which correspond to the Lagrange multipliers for the respective related constraints. Finally, we emphasize such a convex dual formulation obtained may be applied to a large class of similar models in the calculus of variations, including models in elasticity and phase transition.

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MSC: 49N15.

1 Introduction

This article develops a duality principle applicable to a large class of models in the calculus of variations. Specifically in this text, in a first step, we present applications to a model in micro-magnetism. It worth mentioning the final dual variational formulation is concave and is obtained through basic tools of convex analysis, optimization and concerning duality theory.

We emphasize that such results on duality theory here addressed and developed are inspired mainly in the original approaches of J.J.Telega, W.R. Bielski and co-workers presented in the articles (Bielski et al., 1988; Bielski and Telega, 1985; Telega, 1989; Galka and Telega, 1995). Other main reference is the article by Toland (1979).

Details on theoretical results in micro-magnetism may be found in Kronmuller and Fahnle (2003).

Moreover, details on the Sobolev spaces involved may be found in Adams and Fournier (2003).

Similar results and models are addressed in Botelho (2014, 2020, 2009, 2011, 2012).

Basic results on convex analysis are addressed in Rockafellar (1970). Other similar results and approaches may be found in Botelho (2023a); Attouch et al. (2006); Botelho (2023b).

Now, we start to describe the primal variational formulation for the model in micro-magnetism in question.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V_1 \times V_2 \to \mathbb{R}$ defined by

$$\begin{aligned} J(m,f) &= \frac{\alpha}{2}\int_\Omega |\nabla m|^2\,dx + \varphi(m) - \int_\Omega H\cdot m\,dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^3} |f|^2\,dx \end{aligned} \tag{1}$$

where $\alpha > 0$, $m = (m_1, m_2, m_3) \in V_1 = W^{1,2}(\Omega;\mathbb{R}^3)$, represents a magnetization field concerning a ferromagnetic sample Ω and

$$f \in V_2 = L^2(\mathbb{R}^3;\mathbb{R}^3)$$

is a vectorial field on $\mathbb{R}^3$.

Moreover, $H : \Omega \to \mathbb{R}^3$ is a known external magnetic field.

Such a functional is considered subject to the following constraints

$$\begin{gathered} |m| = \sqrt{m_1^2 + m_2^2 + m_3^2} = 1, \text{ in } \Omega, \\ \text{Curl } f = \mathbf{0}, \text{ in } \mathbb{R}^3, \\ \text{div } (-f + m\chi_\Omega) = 0, \text{ in } \mathbb{R}^3. \end{gathered}$$

Here we have denoted

$$\chi_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Moreover, for a functional $\varphi : V_1 \to \mathbb{R}$, we assume the following multi-well format

$$\varphi(m) = \int_\Omega \min_{k \in \{1,\cdots,M\}} g_k(m)\,dx,$$

where $g_k : \mathbb{R}^3 \to \mathbb{R}$ is a convex and twice differentiable function $\forall k \in \{1,\cdots,M\}$ for a fixed $M \in \mathbb{N}$.

Here, we also denote

$$B = \left\{ t = \{t_k\} \text{ measurable } : 0 \leq t_k \leq 1,\ \forall k \in \{1,\cdots,M\} \text{ and } \sum_{k=1}^{M} t_k = 1, \text{ in } \Omega \right\}.$$

With such statements and definitions in mind, we define the functional

$$J_1 : V_1 \times V_2 \times B \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\},$$

by

$$\begin{aligned} J_1(m,f,t) &= \frac{\alpha}{2}\int_\Omega |\nabla m|^2\,dx + \int_\Omega \sum_{k=1}^{M} t_k g_k(m)\,dx \\ &\quad - \int_\Omega H \cdot m\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |f|^2\,dx \\ &\quad + Ind_0(m) + Ind_1(m,f) + Ind_2(f), \end{aligned} \tag{3}$$

where

$$Ind_0(m) = \begin{cases} 0, & \text{if } \sum_{k=1}^{3} m_k^2 = 1, \text{ in } \Omega, \\ +\infty, & \text{otherwise,} \end{cases} \tag{4}$$

$$Ind_1(m,f) = \begin{cases} 0, & \text{if } \operatorname{div}(-f + m\chi_\Omega) = 0, \text{ in } \mathbb{R}^3, \\ +\infty, & \text{otherwise,} \end{cases} \tag{5}$$

$$Ind_2(f) = \begin{cases} 0, & \text{if } \operatorname{Curl} f = \mathbf{0}, \text{ in } \mathbb{R}^3, \\ +\infty, & \text{otherwise.} \end{cases} \tag{6}$$

2 The Main Duality Principle

Denoting $Y = Y^* = L^2(\Omega)$ and $Y_1 = Y_1^* = L^2(\Omega;\mathbb{R}^3)$, we start by defining the functionals $F_1 : V_1 \times B \to \mathbb{R}$ and $F_2 : V_2 \to \mathbb{R}$ by

$$\begin{aligned} F_1(m,t) &= \frac{\alpha}{2}\int_\Omega |\nabla m|^2\,dx + \int_\Omega \sum_{k=1}^{M} t_k g_k(m)\,dx \\ &\quad - \int_\Omega H \cdot m\,dx, \end{aligned} \tag{7}$$

$$F_2(f) = \frac{1}{2}\int_{\mathbb{R}^3} |f|^2\,dx.$$

Thus, already including the Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3$ for the respective concerning constraints, we obtain

$$\begin{aligned} J_1(m,f,t) &= F_1(m,t)+F_2(f) \\ &\quad +Ind_0(m)+Ind_1(m,f)+Ind_2(f) \\ &\geq F_1(m,t)+F_2(f) \\ &\quad +\left\langle \sum_{k=1}^3 m_k^2-1,\lambda_3 \right\rangle_{L^2} \\ &\quad +\langle \text{ div } (-f+m\chi_\Omega),\lambda_2\rangle_{L^2(\mathbb{R}^3)}+\langle \text{ Curl } f,\lambda_1\rangle_{L^2(\mathbb{R}^3)}, \\ &\geq \inf_{m\in V_1}\left\{F_1(m,t)+\left\langle \sum_{k=1}^3 m_k^2-1,\lambda_3\right\rangle_{L^2}-\langle m,\nabla\lambda_2\rangle_{L^2}\right\} \\ &\quad +\inf_{f\in V_2}\left\{\langle f,\nabla\lambda_2+\text{ Curl }\lambda_1\rangle_{L^2(\mathbb{R}^3)}+F_2(f)\right\} \\ &= -F_1^*(\lambda_2,\lambda_3,t)-F_2^*(\lambda_2,\lambda_1), \end{aligned} \tag{8}$$

$\forall \lambda=(\lambda_1,\lambda_2,\lambda_3)\in A^*$, $t\in B$, where

$$A^*=\{\lambda\in Y_1^*\times Y^*\times Y^* \,:\, -\alpha\nabla^2+\lambda_3>\mathbf{0} \text{ and } \lambda_2=0, \text{ on } \partial\Omega\}.$$

Here we have defined $F_1^*: Y^*\times Y^*\times B\to\mathbb{R}$ and $F_2^*: Y^*\times Y_1^*\to\mathbb{R}$ by

$$F_1^*(\lambda_2,\lambda_3,t)=\sup_{m\in V_1}\left\{-F_1(m,t)-\left\langle \sum_{k=1}^3 m_k^2-1,\lambda_3\right\rangle_{L^2}+\langle m,\nabla\lambda_2\rangle_{L^2}\right\},$$

and

$$\begin{aligned} F_2^*(\lambda_2,\lambda_1) &= \sup_{f\in V_2}\left\{-\langle f,\nabla\lambda_2+\text{ Curl }\lambda_1\rangle_{L^2(\mathbb{R}^3)}-F_2(f)\right\} \\ &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\lambda_2+\text{ Curl }\lambda_1|^2\,dx. \end{aligned} \tag{9}$$

Define now $J^*: A^*\times B\to\mathbb{R}$ by

$$J^*(\lambda,t)=J^*(\lambda_1,\lambda_2,\lambda_3,t)=-F_1^*(\lambda_2,\lambda_3,t)-F^*(\lambda_2,\lambda_1).$$

Observe that, from the previous lines, we have obtained

$$\begin{aligned} &\inf_{(m,f)\in V_1\times V_2}\{J(m,f)+Ind_0(m)+Ind_1(m,f)+Ind_2(f)\} \\ =& \inf_{(m,f,t)\in V_1\times V_2\times B} J_1(m,f,t) \\ \geq& \inf_{t\in B}\left\{\sup_{\lambda\in A^*} J^*(\lambda,t)\right\} \\ \geq& \sup_{\lambda\in A^*}\left\{\inf_{t\in B} J^*(\lambda,t)\right\}. \end{aligned} \tag{10}$$

We recall the constraints

$$0 \leq t_k \leq 1, \text{ in } \Omega, \forall k \in \{1, \cdots, M\}$$

are equivalent to

$$t_k^2 - t_k \leq 0, \text{ in } \Omega, \forall k \in \{1, \cdots, M\}.$$

Thus, already including the related Lagrange multipliers, we define the Lagrangian L by

$$L(\lambda, t, \tilde{\lambda}) = J^*(\lambda, t) + \sum_{K=1}^{M} \left\langle t_k^2 - t_k, \left(\tilde{\lambda}_k^0\right)^2 \right\rangle_{L^2} + \left\langle \sum_{k=1}^{M} t_k - 1, \tilde{\lambda}_1 \right\rangle_{L^2}.$$

Let $(\hat{\lambda}, t_0, \tilde{\lambda}) \in A^* \times B \times L^2(\Omega; \mathbb{R}^{M+1})$ be such that

$$\delta L(\hat{\lambda}, t_0, \tilde{\lambda}) = \mathbf{0}.$$

Observe that, defining

$$H_1(m, \hat{\lambda}_2, \hat{\lambda}_3, t_0) = -F_1(m, t_0) - \left\langle \sum_{k=1}^{3} m_k^2 - 1, \hat{\lambda}_3 \right\rangle_{L^2} + \langle m, \nabla \hat{\lambda}_2 \rangle_{L^2},$$

we have that

$$F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0) = \sup_{m \in V_1} \{H_1(m, \hat{\lambda}_3, \hat{\lambda}_2, t_0)\} = H_1(\hat{m}, \hat{\lambda}_2, \hat{\lambda}_3, t_0),$$

where $\hat{m} \in V_1$ is such that

$$\frac{\partial H_1(\hat{m}, \hat{\lambda}_2, \hat{\lambda}_3, t_0)}{\partial m_k} = \mathbf{0},$$

$\forall k \in \{1, 2, 3\}$.

Observe also that symbolically, we may write

$$\frac{\partial F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0)}{\partial \lambda_2} = -\text{ div } \left(\frac{\partial F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0)}{\partial (\nabla \lambda_2)} \right).$$

On the other hand, also symbolically, we have

$$\begin{aligned} \frac{\partial F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0)}{\partial (\nabla \lambda_2)} &= \frac{\partial H_1(\hat{m}, \hat{\lambda}_3, \hat{\lambda}_2, t_0)}{\partial (\nabla \lambda_2)} \\ &\quad + \frac{\partial H_1(\hat{m}, \hat{\lambda}_3, \hat{\lambda}_2, t)}{\partial m} \frac{\partial \hat{m}}{\partial (\nabla \lambda_2)} \\ &= \frac{\partial H_1(\hat{m}, \hat{\lambda}_3, \hat{\lambda}_2, t_0)}{\partial (\nabla \lambda_2)} \\ &= \hat{m}. \end{aligned} \tag{11}$$

In summary, we have got

$$\frac{\partial F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0)}{\partial \lambda_2} = -\text{ div } \hat{m}.$$

Similarly, we may obtain

$$\frac{\partial F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0)}{\partial \lambda_3} = -\left(\sum_{k=1}^{3} \hat{m}_k^2 - 1\right).$$

From such a results and the variation of L in λ_3 we obtain the following extremal equation

$$-\frac{\partial F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0)}{\partial \lambda_3} = \left(\sum_{k=1}^{3} \hat{m}_k^2 - 1\right) = 0, \text{ in } \Omega.$$

From the extremal equation

$$\frac{\partial L(\hat{\lambda}, t_0, \tilde{\lambda})}{\partial \lambda_2} = \mathbf{0}$$

we obtain

$$-\frac{\partial F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0)}{\partial \lambda_2} - \frac{\partial F_2^*(\hat{\lambda}_2, \hat{\lambda}_1)}{\partial \lambda_2} = \mathbf{0},$$

so that

$$\text{div}\,(\hat{m}\chi_\Omega) + \text{ div}\,(\nabla \hat{\lambda}_2 + \text{ Curl } \hat{\lambda}_1) = 0, \text{ in } \mathbb{R}^3.$$

Thus, denoting

$$\hat{f} = -\nabla \hat{\lambda}_2 - \text{ Curl } \hat{\lambda}_1,$$

we have got

$$\text{div}\,(\hat{m}\chi_\Omega) - \text{ div } \hat{f} = 0, \text{ in } \mathbb{R}^3.$$

From the variation of L in λ_1, we may obtain

$$\frac{\partial F_2^*(\hat{\lambda}_2, \hat{\lambda}_1)}{\partial \lambda_1} = \mathbf{0}$$

so that

$$\text{Curl }(\nabla \hat{\lambda}_2 + \text{ Curl } \hat{\lambda}_1) = \mathbf{0},$$

that is,

$$\text{Curl } \hat{f} = \mathbf{0}, \text{ in } \mathbb{R}^3.$$

From such results, we have obtained

$$Ind_0(\hat{m}) = 0, Ind_1(\hat{m}, \hat{f}) = 0 \text{ and } Ind_2(\hat{f}) = 0.$$

Moreover, from the standard Legendre transform properties, we may also obtain

$$F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0) = -F_1(\hat{m}, t_0) - \left\langle \sum_{k=1}^{3} \hat{m}_k^2 - 1, \hat{\lambda}_3 \right\rangle_{L^2} + \langle \hat{m}, \nabla \hat{\lambda}_2 \rangle_{L^2},$$

and

$$F_2^*(\hat{\lambda}_2, \hat{\lambda}_1) = -F_2(\hat{f}) - \langle \hat{f}, \nabla \hat{\lambda}_2 + \text{ Curl } \hat{\lambda}_1 \rangle_{L^2(\mathbb{R}^3)}.$$

Combining these last two equations, we have obtained

$$\begin{aligned} J^*(\hat{\lambda},t_0) &= -F_1^*(\hat{\lambda}_2,\hat{\lambda}_3,t_0)-F_2^*(\hat{\lambda}_2,\hat{\lambda}_1) \\ &= F_1(\hat{m},t_0)+F_2(\hat{f}) \\ &= F_1(\hat{m},t_0)+F_2(\hat{f}) \\ &\quad +Ind_0(\hat{m})+Ind_1(\hat{m},\hat{f})+Ind_2(\hat{f}) \\ &= J_1(\hat{m},\hat{f},t_0). \end{aligned} \tag{12}$$

Suppose $t_0 \in B$ is such that

$$J^*(\hat{\lambda},t_0)=\inf_{t\in B} J^*(\hat{\lambda},t).$$

From an evident concavity of J^* in λ, we have

$$J^*(\hat{\lambda},t_0)=\sup_{\lambda\in A^*} J^*(\lambda,t_0).$$

From such results and a Standard Saddle Point Theorem, we may infer that

$$\begin{aligned} J^*(\hat{\lambda},t_0) &= \sup_{\lambda\in A^*}\left\{\inf_{t\in B} J^*(\lambda,t)\right\} \\ &= \inf_{t\in B}\left\{\sup_{\lambda\in A^*} J^*(\lambda,t)\right\}. \end{aligned} \tag{13}$$

Joining the pieces, we have got

$$\begin{aligned} J(\hat{m},\hat{f}) &= J(\hat{m},\hat{f})+Ind_0(\hat{m})+Ind_1(\hat{m},\hat{f})+Ind_2(\hat{f}) \\ &= \inf_{(m,f)\in V_1\times V_2}\{J(m,f)+Ind_0(m)+Ind_1(m,f)+Ind_2(f)\} \\ &= J_1(\hat{m},\hat{f},t_0) \\ &= \inf_{(m,f,t)\in V_1\times V_2\times B} J_1(m,f,t) \\ &= J^*(\hat{\lambda},t_0) \\ &= \sup_{\lambda\in A^*}\left\{\inf_{t\in B} J^*(\lambda,t)\right\} \\ &= \inf_{t\in B}\left\{\sup_{\lambda\in A^*} J^*(\lambda,t)\right\}. \end{aligned} \tag{14}$$

The objective of this section is complete.

3 Conclusion

In this article, we have developed a duality principle and a related convex dual variational formulation for a model in micro-magnetism. We emphasize such a dual variational formulation is concave in its main variables, which correspond to the Lagrange multipliers related to the respective constraints for the primal functional.

We also highlight the results here obtained are applicable to a large class of models in the calculus of variations, including some plate and shell non-linear theories, models in superconductivity and phase transition, among many others.

In a near future research we intend to apply such results to some of these mentioned related models.

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Competing Interests

Author has declared that no competing interests exist.

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