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# On parametric equivalence, isomorphism and uniqueness: Cycle related graphs

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**Abstract:** This furthers the notions of parametric equivalence, isomorphism and uniqueness in graphs. Results for certain cycle related graphs are presented. Avenues for further research are also suggested.

**Keywords:** Parametric equivalence; Parametric isomorphism; Parametric uniqueness.

**MSC:** 05C12; 05C38; 05C69.

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## 1. Introduction

Unless stated otherwise, graphs will be finite, undirected and connected simple graphs. A shortest path having end vertices $u$ and $v$ is denoted by, $u - v_{(in\ G)}$. If $d_G(u,v) \geq 2$ then a vertex $w$ on $u - v_{(in\ G)}$, $w \neq u$, $w \neq v$ is called an internal vertex on $u - v_{(in\ G)}$. When the context is clear the notation such as $d_G(u,v)$, $deg_G(v)$ can be abbreviated to $d(u,v)$, $deg(v)$ and so on. Good references to important concepts, notation and graph parameters can be found in [1–3].

The notions of parametric equivalence, isomorphism and uniqueness had been introduced in [4]. For ease of reference we recall from [4] as follows: Let $\rho$ denote some minimum or maximum graph parameter related to subsets $V(G)$ of graph $G$. Vertex subsets $X$ and $Y$ is said to be *parametric equivalent* or *$\rho$-equivalent* if and only if both $X$, $Y$ satisfy the parametric conditions of $\rho$. This relation is denoted by $X \equiv_\rho Y$. Furthermore, if $X \equiv_\rho Y$ and the induced graphs $\langle V(G)\backslash X\rangle \cong \langle V(G)\backslash Y\rangle$ then $X$ and $Y$ are said to be *parametric isomorphic*. This isomorphic relation is denoted by $X \cong_\rho Y$. Let all possible vertex subsets of graph $G$ which satisfy $\rho$ be $X_1, X_2, X_3, \dots, X_k$. If $X_1 \cong_\rho X_2 \cong_\rho X_3 \cong_\rho \cdots \cong_\rho X_k$ then $X_i$, $1 \leq i \leq k$ are said to be *parametric unique* or *$\rho$-unique*. The graph $G$ is said to have a *parametric unique* or *$\rho$-unique* solution (or parametric unique $\rho$-set). If $G$ has a unique (exactly one) $\rho$-set $X$, then $X$ is a parametric unique $\rho$-set.

This paper furthers the introductory research presented in [4].

## 2. Confluence in graphs

Shiny *et al.*, [5] introduced the concept of a confluence set (a subset of vertices) of a graph $G$, also see [6] for results on certain derivative graphs. Recall that for a non-complete graph $G$, a non-empty subset $\mathcal{X} \subseteq V(G)$ is said to be a confluence set if for every unordered pair $\{u,v\}$ of distinct vertices (if such exist) in $V(G)\backslash\mathcal{X}$ for which $d_G(u,v) \geq 2$ there exists at least one $u - v_{(in\ G)}$ with at least one internal vertex, $w \in \mathcal{X}$. Also a vertex $u \in \mathcal{X}$ is called a *confluence vertex* of $G$. A minimal confluence set $\mathcal{X}$ (also called a $\zeta$-set) has no proper subset which is a confluence set of $G$. The cardinality of a minimum confluence set is called the *confluence number* of $G$ and is denoted by $\zeta(G)$. A minimal confluence set is denoted by $\mathcal{C}$. To distinguish between different graphs the notation $\mathcal{C}_G$ may be used for a minimum confluence set of $G$. We recall two important results from [4]. We remind that for a complete graph the confluence number is 0 hence, $\mathcal{C}_{K_n} = \emptyset$, $n \geq 1$.

**Proposition 1.** [4] *A path $P_n$ has a parametric unique $\zeta$-set if and only if $n = 1, 2$ or $n = 4 + 3i$ or $n = 5 + 3i$, $i = 0, 1, 2, \dots$.*

**Proposition 2.** [4] *A cycle $C_n$ has a parametric unique $\zeta$-set if and only if $n = 3,4$ or $n = 5+3i$ or $n = 6+3i$, $i = 0,1,2,\ldots$.*

### 2.1. Cycle related graphs

Henceforth, a cycle $C_n$, $n \geq 3$ of order $n$ has the vertex set $V(C_n) = \{v_i : i = 1,2,3,\ldots,n\}$.

(a) A wheel graph (simply, a wheel) $W_n$ is obtained from a cycle $C_n$, $n \geq 3$ with an additional central vertex $v_0$ and the additional edges $v_0v_1$, $1 \leq i \leq n$. The cycle is called the *rim* and the edges $v_0v_i$, $1 \leq i \leq n$ are called *spokes*. Alternatively, $W_n = C_n + K_1$ and $V(K_1) = \{v_0\}$.

**Proposition 3.** *A wheel graph $W_n$ has a parametric unique $\zeta$-set.*

**Proof.** Since $W_3$ is complete the result is trivial. For $n \geq 4$ the distance $d(v_i, v_j) \leq 2$ for all distinct pairs. For $i, j \neq 0$ and $v_i$ not adjacent to $v_j$ there exists a 3-path (or 2-distance path) with $v_0$ the internal vertex. Hence, the unique $\zeta$-set is $\{v_0\}$, therefore parametric unique. □

(b) A helm graph $H_n$ is obtained from a wheel graph $W_n$ by adding a pendent vertex (or leaf) $u_i$ to each rim vertex $v_i$.

**Proposition 4.** *(a) The helm graph $H_3$ does not have a parametric unique $\zeta$-set.*
*(b) A helm graph $H_n$, $n \geq 4$ has a parametric unique $\zeta$-set.*

**Proof.** (a) Consider $H_3$. Clearly and without loss of generality the sets $X_1 = \{v_0, v_1, v_2\}$, $X_2 = \{v_1, v_2, v_3\}$ and $X_3 = \{v_1, v_2, u_3\}$ are all minimal confluence sets. Hence $\zeta(H_3) \leq 3$. It is easy to verify that no 2-vertex subset is a confluence set. Thus, $\zeta(H_3) > 2$. Also, $\langle V(H_3)\backslash X_1\rangle \ncong \langle V(H_3)\backslash X_2\rangle$. Therefore $H_3$ does not have a parametric unique $\zeta$-set. The aforesaid follows in essence from the fact that $H_3$ is complete. Therefore, it is not necessary for $v_0$ to be in all $\zeta$-sets.

(b) For $H_n$, $n \geq 4$ the distance $d(u_i, u_{i+1}) = 3$ hence a rim vertex is required. The distance $d(u_i, u_{i+2}) = 5$ hence the vertex $v_0$ will suffice along the 5-path $u_iv_iv_0v_{i+2}u_{i+2}$. By symmetry considerations and therefore up to isomorphism and without loss of generality we have two subcases.

**Subcase 1.** If $n$ is even the set $X_1 = \{v_0, v_1, v_3, v_5, \ldots, v_{n-1}\}$ is a $\zeta$-set and clearly $H_n$ has a parametric unique $\zeta$-set.

**Subcase 2.** If $n$ is odd the sets $X_1 = \{v_0, v_1, v_3, v_5, \ldots, v_{n-1}\}$ and $X_2 = \{v_0, v_1, v_3, v_5, \ldots, v_{n-2}, v_n\}$ are a $\zeta$-sets. Clearly $\langle V(H_n)\backslash X_1\rangle \cong \langle V(H_n)\backslash X_1\rangle$. Thus $H_n$ has a parametric unique $\zeta$-set.

□

As a direct consequence of the proof of Proposition 4, we get the next corollary.

**Corollary 1.** *A helm graph has $\zeta(H_n) = \lceil \frac{n}{2} \rceil + 1$.*

(c) A flower graph $Fl_n$ is obtained from a helm graph $H_n$ by adding the edges $v_0u_i$, $1 \leq i \leq n$.

**Proposition 5.** *A flower graph $Fl_n$ has a parametric unique $\zeta$-set.*

**Proof.** The result follows by similar reasoning as in the proof of Proposition 3. □

As a direct consequence of Proposition 5, we get the next corollary.

**Corollary 2.** *A flower graph has $\zeta(Fl_n) = 1$.*

(d) A closed helm graph $H_n^c$ is obtained from a helm graph $H_n$ by completing a cycle, $C'_n = u_1u_2u_3\cdots u_nu_1$ on the leafs of $H_n$.

**Proposition 6.** *(a) A closed helm graph $H_n^c$ for $n = 4$ or $n$ is odd does not have a parametric unique $\zeta$-set.*
*(b) A closed helm graph $H_n^c$, $n \geq 6$ and even, has a parametric unique $\zeta$-set.*

**Proof.** It is easy to verify that all distance paths such that $d(u_i, u_j) \leq 3$ are paths on $C'_n$. Also, for $u_i, u_j \in \mathcal{C}_{C'_n}$ we have $d(u_i, u_j) \leq 3$. It follows that $\mathcal{C}_{C'_n} \subseteq \mathcal{C}_{H^c_n}$.

(a) By similar reasoning to that in the proof of Proposition 4(a) it follows that $H^c_3$ and $H^c_4$ do not have a unique $\zeta$-set.

From the set $X_1 = \{v_i : u_i \notin \mathcal{C}_{C'_n}\} \cup \{v_0\}$ it is possible to select a minimum confluence set in respect of the spanning subgraph $H_n$ say set $X_2$. The set $\mathcal{C}_{H^c_n} = \mathcal{C}_{C'_n} \cup X_2$ is a minimum confluence set.

**Subcase (a)(1)**. Since by symmetry the choice of say, $X_2$ can be fixed, For $n \geq 5$ and odd, the choice of $\mathcal{C}_{C'_n}$ can rotate such that $\langle V(H^c_n) \backslash \mathcal{C}_{H^c_n} \rangle$ does not remain isomorphic.

(b) By similar reasoning $X_2$ can be fixed. However, for $n \geq 6$ and even and by symmetry properties of $C'_n$ all choices of $\mathcal{C}_{C'_n}$ yield isomorphic $\langle V(H^c_n) \backslash \mathcal{C}_{H^c_n} \rangle$.

□

As a direct consequence of the proof of Proposition 6, we get the next corollary.

**Corollary 3.** *A closed helm graph has* $\zeta(H^c_n) = \lceil \frac{n}{2} \rceil + 1$.

(e) A gear graph $G_n$ is obtain from a wheel graph $W_n$ by inserting a vertex $u_i$ on the edge $v_i v_{i+1}$ and $n+1 \equiv 1$. Note that $G_n$ has $2n+1$ vertices and $3n$ edges. The *rim* is now called a *boundary cycle* denoted by $C^b(G_n)$.

**Proposition 7.** *(a) $G_3$ has a parametric unique $\zeta$-set.*
*(b) A gear graph $G_n$ and $n \geq 5$ is odd does not have a parametric unique $\zeta$-set.*
*(c) A gear graph $G_n$ and $n \geq 4$ is even has a parametric unique $\zeta$-set.*

**Proof.** (a) For $G_3$ it follows easily that up to isomorphism the $\zeta$-set $\{u_1, v_3\}$ is unique.

(b) The inner-area enclosed by the cycle $C'_{2n} = v_1 u_1 v_2 u_2 \cdots v_n u_n v_1$ can be partitioned into $n$ planar areas, each enclosed by a $C_4$. For all pairs $v_i, v_j$ it is necessary and sufficient that $v_0 \in \zeta$-set. Let $n \geq 5$ be odd. Without loss of generality, an optimal minimal confluence set is given by $X_1 = \{v_0, u_1, u_3, \ldots, u_{n-2}, u_{n-1}\}$ or $X_2 = \{v_0, u_1, u_3, \ldots, u_{n-2}, v_n\}$ or $X_3 = \{v_0, u_1, u_3, \ldots, u_{n-2}, u_n\}$. Hence, $\zeta(G_n) \leq \lceil \frac{2n}{4} \rceil + 1 = \lceil \frac{n}{2} \rceil + 1$. Because the boundary cycle $C^b(G_n)$ has $\zeta(C^b(G_n)) = \lceil \frac{2n}{3} \rceil$ it follows that $\zeta(G_n) \geq \lceil \frac{2n}{3} \rceil$. However for $n$ is odd, $\lceil \frac{2n}{3} \rceil = \lceil \frac{n}{2} \rceil + 1$. Since,

$$\langle V(G_n) \backslash X_1 \rangle \ncong \langle V(G_n) \backslash X_2 \rangle.$$

It follows that a gear graph $G_n$ does not have a parametric unique $\zeta$-set for $n$ is odd.

(c) For $n \geq 4$ and even, reasoning similar to that in (b) show that up to isomorphism the $\zeta$-set $X_1 = \{v_0, u_1, u_3, \ldots, u_{n-2}, u_{n-1}\}$ is unique. Reasoning in respect of bounds on $\zeta(G_n)$ similar to that in (a) settles the result.

□

As a direct consequence of the proof of Proposition 7, we get the next corollary.

**Corollary 4.** *The gear graph $G_3$ has $\zeta(G_3) = 2$. A gear graph of order $n \geq 4$ has $\zeta(G_n) = \lceil \frac{n}{2} \rceil + 1$.*

(f) A sun graph $S^{\boxtimes}_n$, $n \geq 3$ is obtained by taking the complete graph $K_n$ on the vertices $v_1, v_2, v_3, \ldots, v_n$ together the isolated vertices $u_i$, $1 \leq i \leq n$ and adding the edges $v_i u_i$, $u_i v_{i+1}$ and $n+1 \equiv 1$. The boundary cycle of a sun graph is the cycle $C^b(S^{\boxtimes}_n) = v_1 u_1 v_2 u_2 v_3 u_3 \cdots u_n v_1$.

**Proposition 8.** *A sun graph $S^{\boxtimes}_n$, $n \geq 3$ has a parametric unique $\zeta$-set if and only if $C^b(S^{\boxtimes}_n)$ is of order $n = 3i$, $i = 1, 2, 3, \ldots$*

**Proof.** Since all pairs $v_i, v_j$ are adjacent it suffices to only consider a $\zeta$-set of $C^b(S^{\boxtimes}_n)$. Since $deg(u_i) = 2$ and $deg(v_j) = 3$ any $\zeta$-set must be graphically symmetrical for a sun graph to have a parametric unique $\zeta$-set. A graphically symmetrical $\zeta$-set means that, measured along the boundary cycle, $min\{d(v_j, u_k) : v_j, u_k \in \zeta\text{-set}\} = 3$. It implies that $n = 3i$, $i = 1, 2, 3, \ldots$.

The converse follows from the fact that sun graphs with $C^b(S_n^{\boxtimes})$ of order $n \neq 3i$, $i = 1,2,3,\ldots$ do not have graphically symmetrical $\zeta$-sets of even order. □

Note that if a sun graph has a parametric unique $\zeta$-set then $\zeta(S_n^{\boxtimes})$ is even. Furthermore, as a direct consequence of the proof of Proposition 8, we get the next corollary.

**Corollary 5.** *A sun graph has* $\zeta(S_n^{\boxtimes}) = \lceil \frac{2n}{3} \rceil$.

(g) A sunflower graph $S_n^{\circledast}$, $n \geq 3$ is obtained by taking the wheel graph $W_n$ together the isolated vertices $u_i$, $1 \leq i \leq n$ and adding the edges $v_iu_i$, $u_iv_{i+1}$ and $n+1 \equiv 1$. The boundary cycle of a sun graph is the cycle $C^b(S_n^{\circledast}) = v_1u_1v_2u_2v_3u_3\cdots u_nv_1$.

**Proposition 9.** *A sunflower graph* $S_n^{\circledast}$*,* $n \geq 3$ *does not have a parametric unique* $\zeta$*-set.*

**Proof.** For all pairs $v_i, v_j$ it is sufficient that $v_0 \in \zeta$-set. Thereafter any $\zeta$-set $X_1$ in respect of $C^b(S_n^{\circledast})$ is required to obtain $\mathcal{C}_{S_n^{\circledast}} = X_1 \cup \{v_0\}$. It implies that $\zeta(S_n^{\circledast}) = n$. In turn, the aforesaid confluence number permits that say, $X_2 = \{v_1, v_2, v_3, \ldots, v_n\}$ or $X_3 = \{v_1, v_2, v_3, \ldots, v_{n-1}, u_{n-1}\}$ are $\zeta$-sets. Since, $\langle V(S_n^{\circledast})\backslash X_1\rangle \ncong \langle V(S_n^{\circledast})\backslash X_2\rangle \ncong \langle V(S_n^{\circledast})\backslash X_3\rangle$ the result follows. □

As a direct consequence of the proof of Proposition 9, we get the next corollary.

**Corollary 6.** *A sunflower graph has* $\zeta(S_n^{\circledast}) = n$.

(h) A sunlet graph $S_n^{\ominus}$, $n \geq 3$ is obtained by taking cycle $C_n$ together the isolated vertices $u_i$, $1 \leq i \leq n$ and adding the pendent edges $v_iu_i$.

**Proposition 10.** *A sunlet graph* $S_n^{\ominus}$*,* $n \geq 3$ *has a parametric unique* $\zeta$*-set.*

**Proof. Case 1.** Let $n \geq 3$ and odd. Without loss of generality and by isomorphism, it is easy to verify that the sets $X_1 = \{v_1, v_3, v_5, \ldots, v_n\}$ and $X_2 = \{v_1, v_3, v_5, \ldots, v_{n-2}, v_{n-1}\}$ are $\zeta$-sets. Furthermore, up to isomorphism those are the only distinguishable $\zeta$-sets. Since,

$$\langle (V(S_n^{\ominus})\backslash X_1\rangle \cong \langle (V(S_n^{\ominus})\backslash X_2\rangle,$$

the result follows for $n \geq 3$ and odd.
**Case 2.** By similar reasoning as in Case 1 the result follows for $n \geq 4$ and even. □

As a direct consequence of the proof of Proposition 10, we get the next corollary.

**Corollary 7.** *A sunlet graph has* $\zeta(S_n^{\ominus}) = \lceil \frac{n}{2} \rceil$.

(i) A circular ladder (or prism graph) $L_n^{\circ}$, $n \geq 3$ is obtained by taking two cycles of equal order $n$. Label as, $C_n^1 = v_1v_2v_3\cdots v_nv_1$ and $C_n^2 = u_1u_2u_3\cdots u_nu_1$. Add the edges $v_iu_i$, $1 \leq i \leq n$. A circular ladder can be viewed as $H_n^c - v_0$.

**Proposition 11.** *A circular ladder graph* $L_n^{\circ}$ *has a parametric unique* $\zeta$*-set if and only if* $n = 4$ *or* $n = 3i$ *for* $i = 2,3,4,\ldots$.

**Proof. Part 1.** For $n = 4$, $X_i = \{u_i, v_j\}$, $i = 1,2,3,4$, $j \in \{1,2,3,4\}$ such that $d(u_i, v_j) = 3$, are the minimum confluence sets for $L_4^{\circ}$. Since $\langle V(L_4^{\circ})\backslash X_i\rangle$ are $C_6$ for $i = 1,2,3,4$, we have the result for $n = 4$.

In a circular ladder graph $L_n^{\circ}$, $n \neq 4$ there are $n$ copies of $C_4 = v_iu_iu_{i+1}v_{i+1}$. For each $C_4 = v_iu_iu_{i+1}v_{i+1}$, at least one of the vertices $v_i, u_i, u_{i+1}, v_{i+1}$ belongs to every minimum confluence set of $L_n^{\circ}$.
**Part 2.** For $n = 3$, $X_1 = \{v_1, v_2\}$ and $X_2 = \{v_1, u_2\}$ are two minimum confluence set for $L_3^{\circ}$. However, $\langle V(L_3^{\circ})\backslash X_1\rangle$ and $\langle V(L_3^{\circ})\backslash X_2\rangle$ are not isomorphic. Hence $L_3^{\circ}$ has no unique parametric set.
**Part 3.** For $n = 3i$, $i = 2,3,..,$ let $\mathcal{C}_{C_n}(v_i)$ be a minimum confluence set of $C_n$ starting from $v_i$ and $\mathcal{C}_{C_n'}(u_j)$ be a minimum confluence set of $C_n'$ starting from $u_j$. Then for $i \neq j, X_{ij} = \mathcal{C}_{C_n}(v_i) \cup \mathcal{C}_{C_n'}(u_j)$ is a minimum confluence

set for $L_n^\circ$ and $\langle V(L_n^\circ)\backslash X_{ij}\rangle$ consists of $\frac{n}{3}$ copies of $P_3$. Hence the result for $n = 3i$, $i = 2, 3, ...$

**Part 4.** If $n \equiv 2(mod\ 3)$. Let $X_1$ be the minimum confluence set for $L_n^\circ$ such that $u_i, u_{i+2}, v_{i+1} \in X_1$ and let $X_2$ be the minimum confluence set for $L_n^\circ$ such that $u_i, u_{i+2}, v_i \in X_2$. Then $\langle V(L_n^\circ)\backslash X_1\rangle$ and $\langle V(L_n^\circ)\backslash X_2\rangle$ are not isomorphic. Hence $L_n^\circ$ has no parametric unique set if $n_{\geq 5} \equiv 2(mod\ 3)$.

By a similar argument we have to prove that $L_n^\circ$ has no parametric unique set if $n_{\geq 7} \equiv 1(mod\ 3)$.

Since all $n \in \mathbb{N}_{\geq 3}$ have been accounted for the 'if' has been settled.

For all valid cases the converse, 'only if', follows through reasoning by contradiction. □

**Corollary 8.** *A circular ladder has,*

$$\zeta(L_n^\circ) = \begin{cases} 2, & \text{if } n = 4; \\ 2\lceil \frac{n}{3} \rceil, & \text{if } n = 3 \text{ or } n \geq 5. \end{cases}$$

**Proof.** The result is a consequence of the proof of Proposition 11. The exception lies in the fact that $L_4^\circ$ has $5 = n_{=4} + 1$ cycles $C_4$ to account for. All other $L_{n_{\neq 4}}^\circ$ have $n$ cycles $C_4$ to account for. □

Observe that the confluence number of a circular ladder is always even.

(j) A tadpole graph $T(m, n)$, $m \geq 3$, $n \geq 1$ is obtained from a cycle $C_m = v_1v_2v_3 \cdots v_mv_1$ and a path $P_n = u_1u_2u_3 \cdots u_n$ by adding an edge between an end-vertex of $P_n$ and a vertex of $C_m$. The new edge is also called a *bridge*.

**Proposition 12.** *A tadpole graph $T(m, n)$, $m \geq 3$, $n \geq 1$:*

*(a) Tadpole graphs $T(3, n)$, $n \geq 1$ have a parametric unique $\zeta$-set if and only if $n = 3i$, $i = 1, 2, 3, \ldots$.*
*(b) Tadpole graphs $T(4, 1)$, $T(4, 2)$ have a parametric unique $\zeta$-sets.*
*(c) Tadpole graphs $T(5, 1)$ does not have a parametric unique $\zeta$-set and $T(5, 2)$ has.*
*(d) Tadpole graphs $T(m, 1)$, $T(m, 2)$, $m \geq 6$ have a parametric unique $\zeta$-set if and only if $m = 6 + 3i$, $i = 0, 1, 2, \ldots$*
*(e) Tadpole graphs $T(m, n)$, $m \geq 4$ and $n \geq 3$ have a parametric unique $\zeta$-set if and only if both the cycle $C_m$ and the path $P_n$ have parametric unique $\zeta$-sets.*
*(f) All other tadpole graphs as excluded through (a) to (f) do not have a parametric unique $\zeta$-set.*

**Proof.** (a) The tadpole graphs $T(3, n)$, $n \geq 1$ does not have a parametric unique $\zeta$-set for $P_1$, $P_2$ (straightforward).

**Subcase (a)(1).** For $n + 2 = 5 + 3i$, $i = 0, 1, 2, \ldots$ the $\zeta$-set of $P_{n+2}$ is unique hence, $T(3, n)$ has a parametric unique $\zeta$-set.

**Subcase (a)(2).** For $n + 2 = 6 + 3i$, $i = 0, 1, 2, \ldots$ the $\zeta$-set of $P_{n+2}$ is not parametric unique hence, $T(3, n)$ does not have a parametric unique $\zeta$-set.

**Subcase (a)(3).** For $n + 2 = 7 + 3i$, $i = 0, 1, 2, \ldots$ the $\zeta$-set of $P_{n+2}$ is parametric unique. However, since some $\zeta$-sets may contain vertex $v_j$ of the bridge the tadpole $T(3, n)$ does not have a parametric unique $\zeta$-set.

All tadpoles $T(3, n)$, $n \geq 1$ have been accounted for because,

$$\mathbb{N} = \{1, 2\} \cup \{3 + 3i : i = 0, 1, 2, \ldots\} \cup \{4 + 3i : i = 0, 1, 2, \ldots\} \cup \{5 + 3i : i = 0, 1, 2, \ldots\}.$$

(b) The tadpole graphs $T(4, n)$, $n \geq 1$ have a parametric unique $\zeta$-set for $P_1$, $P_2$. It follows from the fact that a bridge vertex say, $v_i$ has to be in any $\zeta$-set.
Subcases $n + 2 = 5 + 3i$, $n + 2 = 6 + 3i$ and $n + 2 = 7 + 3i$, $i = 0, 1, 2, \ldots$ will be settled in (d) and (e) below.

(c) The tadpole graphs $T(5, n)$, $n \geq 1$ does not have a parametric unique $\zeta$-set for $P_1$ bacause it is easy to verify that an end-vertex of the bridge need not be in all $\zeta$-sets. However for $P_2$ the tadpole has a parametric unique $\zeta$-set. It follows from the fact that a bridge vertex say, $v_i$ has to be in any $\zeta$-set.
Subcases $n + 2 = 5 + 3i$, $n + 2 = 6 + 3i$ and $n + 2 = 7 + 3i$, $i = 0, 1, 2, \ldots$ will be settled in (d) and (e) below.

(d) The tadpoles $T(m, 1)$, $T(m, 2)$, $m \geq 6$ do not require that vertices $u_1$ and/or $u_2$ to necessarily be in a $\zeta$-set. Hence, all $\zeta$-sets of cycle $C_m$ which contain a vertex of the bridge suffice to be $\zeta$-sets of the tadpoles. Therefore has a parametric unique $\zeta$-set if and only if $C_m$ has a unique $\zeta$-set. Therefore, if and only if $m = 6 + 3i$, $i = 0, 1, 2, \ldots$ The converse follows easily by contradiction.

(e) Finally, for a tadpole $T(m,n)$, $m \geq 4$ and $n \geq 3$ and both the cycle $C_m$ and the path $P_n$ have parametric unique $\zeta$-sets, it is easy to verify that the $\zeta$-sets of the tadpole all contain a vertex $v_j$ of the bridge. Therefore the tadpole has a parametric $\zeta$-set. Else, it is always possible to find a $\zeta$-set of the tadpole which contains a vertex $v_j$ which is on the bridge and another $\zeta$-set which does not. Therefore, such tadpoles do not have a parametric unique $\zeta$-set. Hence, the tadpoles $T(m,n)$, $m \geq 4$ and $n \geq 3$ have a parametric unique $\zeta$-set if and only if both $C_m$ and $P_n$ have parametric unique $\zeta$-sets.

(f) All other tadpole graphs which were excluded through reasoning of proof, (a) to (e) do not have a parametric unique $\zeta$-set.

□

(k) A lollipop graph $L^{\boxtimes}(m,n)$, $m \geq 3$, $n \geq 1$ is obtained from a complete graph $K_m$ and a path $P_n$ by adding a bridge between an end-vertex of $P_n$ and a vertex of $C_m$.

**Proposition 13.** *A lollipop graph $L^{\boxtimes}(m,n)$, $m \geq 3$, $n \geq 1$ has a parametric unique $\zeta$-set if and only if $n = 3i$, $i = 1,2,3,\ldots$.*

**Proof.** The proof follows directly from the proof of Proposition 12(a). □

(l) A generalized barbell graph $B(n,m)$, $n,m \geq 3$ is obtained from two complete graph $K_n$, $K_m$ and adding a bridge.

**Proposition 14.** *A generalized barbell graph $B(n,m)$, $n,m \geq 3$ has a parametric unique $\zeta$-set if and only if $n = m$.*

**Proof.** Let $K_n$ be on vertices $v_1, v_2, v_3, \ldots, v_n$ and $K_m$ on vertices $u_1, u_2, u_3, \ldots, u_m$. For any pair $v_iu_j$ and edge $v_iu_j$ not the bridge, the distance $d(v_i,u_j) = 2$ or 3. Therefore any vertex of the bridge yields a $\zeta$-set. Without loss of generality let the $\zeta$-set be $\{v_k\}$. It follows that $\langle V(B(n,m))\backslash\{v_k\}\rangle \cong K_{n-1} \cup K_m$. Hence, $B(n,m)$ has a parametric unique $\zeta$-set if and only if $n = m$. □

## 3. Conclusion

The study of cycle related graphs has not exhausted. Note that for those cycle related graphs which do not have a parametric unique $\zeta$-set the *proof by contradiction* can be utilized well.

The idea of combined parametric conditions remains open. Note that the parametric conditions will be ordered pairs. For example, the path $P_3 = v_1v_2v_3$ has a unique minimum dominating set i.e. the $\gamma$-set $X_1 = \{v_2\}$. Since $X_1$ is also a $\zeta$-set of $P_3$ the set is said to be a parametric unique $(\gamma,\zeta)$-set. However, since $X_1$ *per se* is not a parametric unique $\zeta$-set, it cannot be said to be a parametric unique $(\zeta,\gamma)$-set. On the other hand for a star $S_{1,n}$, $n \geq 3$ the set $X_1 = \{v_0\}$ is both a parametric $(\gamma,\zeta)$-set and a parametric unique $(\zeta,\gamma)$-set. Studying such parametric combinations for say parameters $\rho_1(G)$ and $\rho_2(G)$ requires that, $\rho_1(G) = \rho_2(G)$.

**Conjecture 1.** *If graph $G$ has a pendent vertex then $G$ has a unique $\zeta$-set if and only no $\zeta$-set exists which contains a pendent vertex.*

A strict proof of Corollary 8 through mathematical induction is an interesting exercise for the reader.

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